Problems

This section offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, Room 211-A, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu.*

Proposals IMC-2008

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic and differentiable function. Prove that for any positive integer n there exists a real number ξ such that

$$f(\xi + n) = f(\xi) + nf'(\xi)$$

2. Compute the following determinant

$$\begin{vmatrix} 1 & a_1 & \dots & a_1^{n-2} & (a_2 + a_3 + \dots + a_n)^{n-1} \\ 1 & a_2 & \dots & a_2^{n-2} & (a_1 + a_3 + \dots + a_n)^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & \dots & a_n^{n-2} & (a_1 + a_2 + \dots + a_{n-1})^{n-1} \end{vmatrix}$$

3. Find all real triplets (x, y, z) such that

$$x + y + z = 2,$$

$$2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2} = 6\sqrt[9]{2}.$$

4. Prove that

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\, dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\, dx < \frac{\pi^2}{12}.$$

5. Assume that the function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ is analytic and satisfies $|f(z)| \le \sqrt[3]{|z|} + \dfrac{1}{\sqrt[3]{|z|}}$. Prove that f is constant.

6. Suppose that the zeroes $z_1, z_2, \ldots, z_n$ of polynomial $A(z) = z^n + \sum_{k=1}^{n-1} a_k z^k - a_1$ are distinct nonzero complex numbers. Prove that

$$\sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1 \\ j \ne k}}^{n} \frac{1}{z_k - z_j} = 0$$

(*Proposed by* **J. L. Díaz-Barrero and P. G. Popescu**)