IMC-2009

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1. Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \mathbb{R}$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n = \begin{cases} \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \frac{1}{\alpha}; \\ 0, & n \leq \frac{1}{\alpha} \end{cases}$$

is convergent and determine its limit.

2. Let α be a real number. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \alpha(1 + x^2)\left[1 + \int_0^x \frac{f(t)}{1+t^2}\, dt\right]$$

3. Let R and r denote the radii of the circumcircle and the incircle of a triangle ABC with sides a, b, c and semi-perimeter s. Prove that

$$\frac{(s-a)^4}{c(s-b)} + \frac{(s-b)^4}{a(s-c)} + \frac{(s-c)^4}{b(s-a)} > \frac{3r}{4}\sqrt[3]{\frac{Rs^2}{2}}$$

4. Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ $(a_k \neq 0)$ and $B(z) = z^{n+1} + \sum_{k=0}^{n} b_k z^k$ $(b_k \neq 0)$ be two prime polynomials with roots $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_{n+1}$ respectively. Prove that

$$\frac{A(w_1)A(w_2)\ldots A(w_{n+1})}{B(z_1)B(z_2)\ldots B(z_n)}$$

is an integer and determine its value.

5. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ be the prime factorization of a positive integer m such that the sum of its divisors is $6m$. Prove that

$$\prod_{k=1}^{r}\left(1 + \frac{1}{p_k}\right) \leq 6 < \prod_{k=1}^{r}\left(1 - \frac{1}{p_k}\right)^{-1}$$

6. Let A_k and B_k be the upper left corner submatrices of order k in positive defined matrices A and B of order n, respectively. Prove that

$$|A + B| \geq |A|\left(1 + \sum_{k=1}^{n-1} \frac{|B_k|}{|A_k|}\right) + |B|\left(1 + \sum_{k=1}^{n-1} \frac{|A_k|}{|B_k|}\right),$$

where $|X| = \det(X)$. Recall that a matrix X is defined positive $(X > 0)$ when all its eigenvalues are strictly positive.

7. Let $f:[0,1]\to\mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f\left(\frac{k}{n}\right)$$

8. Let A, B be matrices of the same size. Prove that

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A-B)\det(A+B)$$

9. A certain multiplicative operation on a nonempty set G is associative and allows cancelations on the left, and there exists $a\in G$ such that $x^3 = axa$ for all $x\in G$. Prove that G endowed with this operation is an Abelian group.

10. Consider the set of positive numbers $U=\{1,2,\ldots,6024\}$. Prove that for any partition of U into three classes with 2008 elements one can choose one number for each of the classes such that one of the chosen numbers is the sum of the other two.

11. Let a,b,c,d be positive real numbers. Prove that

$$\frac{ab+bc+ca}{a^3+b^3+c^3}+\frac{ab+bd+da}{a^3+b^3+d^3}+\frac{ac+cd+da}{a^3+c^3+d^3}+\frac{bc+cd+db}{b^3+c^3+d^3}$$
$$\le \min\left\{\frac{a^2+b^2}{(ab)^{3/2}}+\frac{c^2+d^2}{(cd)^{3/2}},\frac{a^2+c^2}{(ac)^{3/2}}+\frac{b^2+d^2}{(bd)^{3/2}},\frac{a^2+d^2}{(ad)^{3/2}}+\frac{b^2+c^2}{(bc)^{3/2}}\right\}$$

12 (*). Let $f_k:[0,1]\to\mathbb{R}, 1\le k\le n$, be any set of real integrable functions in $[0,1]$. Prove that

$$\left|\prod_{k=1}^{n}\int_0^1 f_k(x)\,dx-\int_0^1\prod_{k=1}^{n}f_k(x)\,dx\right|^n\le\prod_{k=1}^{n}\left|\int_0^1 f_k^n(x)\,dx-\left(\int_0^1 f_k(x)\,dx\right)^n\right|$$

14. Let F_n be the n^{th} Fibonacci number defined by $F_0=0, F_1=1$, and $F_n=F_{n-1}+F_{n-2}$ for all $n\ge 2$. Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.

15. Calculate

$$\lim_{n\to\infty}\ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right].$$

16. Let n be a positive integer. Prove that

$$\sin\left(\frac{P_{n+2}}{4P_nP_{n+1}}\right)+\cos\left(\frac{P_{n+2}}{4P_nP_{n+1}}\right)<\frac{3}{2}\sec\left(\frac{3P_n+2P_{n-1}}{4P_nP_{n+1}}\right),$$

where P_n is the n^{th} Pell number defined by $P_0=0$, $P_1=1$ and for $n\ge 2$, $P_n=2P_{n-1}+P_{n-2}$.

17. Let n be a positive integer greater or equal than 2 and let f be a convex function. Prove that

$$\sum_{k=0}^{n}f(k^2)\binom{n}{k}^2\ge\binom{2n}{n}f\left(\frac{n^3}{2(2n-1)}\right).$$

18 . Let X and Y be square matrices of order n. Prove that

$$|X| \cdot |Y| = \sum_{k=1}^{n} |X_k| \cdot |Y_k|,$$

where the matrices X_k and Y_k are obtained from X and Y, respectively, by interchanging their respective first and kth columns.