Proposals

Send solutions to: **José Luis Díaz-Barrero**, Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, Room 211-A, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu*.

---

1. *Let* $A \in M_{2\times 2}(\mathbb{C})$. *Show that* $(trA)^2 = tr(A^2) + 2\det(A)$.

2. *Let* f *be a continuous, nonconstant, real function, and assume the existence of an* F *such that* $f(x+y) = F[f(x), f(y)]$ *for all* $x, y \in \mathbb{R}$. *Show that* f *is strictly monotone.*

3. *Let* $A_1, A_2, \ldots, A_n$ *be the vertices of a regular* $n-$*gon* $\mathcal{P}$ *inscribed in a circle* $\mathcal{C}$ *with radius one. Let* d_k, $1 \le k \le n$, *be the distances from an arbitrary point* P *lying on* $\mathcal{C}$ *to the vertices of* $\mathcal{P}$. *Prove that*

$$\frac{1}{n}\sum_{k=1}^{n} d_k^2$$

is an integer and determine its value.

4. *Let* n *and* k *be positive integers such that* $\sqrt[k]{n}$ *is rational. Show that* n *is a perfect* k^{th} *power.*

5. *Let* G *be a finite group and* K *be a normal subgroup of* G *of order* p, *where* p *is the smallest prime dividing* $|G|$. *Prove that* K *is a subgroup of* $Z(G)$, *where* $Z(G)$ *denotes the center of the group.*

6. *The set* A *consists of nine positive integers, none of which has a prime divisor larger than six. Prove that* A *has two elements whose product is a perfect square.*

7. *Let* A *be the set of positive integers that do not contain the digit* 9 *in their decimal expansions. Show that the series*

$$\sum_{a\in A}\frac{1}{a}$$

is convergent.

8. *Let* u, v, w *be real numbers and let*

$$A(x,y,z) = \sum_{sym}(ux^3 + vx^2y + wxyz)$$

Suppose that $P(1,1,1) \ge 0$, $P(1,1,0) \ge 0$ *and* $P(2,1,1) \ge 0$. *Prove that* $P(a,b,c) \ge 0$, *whenever* a, b, c *are the sides of a triangle.*

9. *Let* A *be a* 3×3 *real orthogonal matrix with* $\det(A) = 1$. *Prove that*

$$(trA - 1)^2 + \sum_{i<j}(a_{ij} - a_{ji})^2 = 4$$

10. *Let* $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *be a polynomial of with complex coefficients of degree* $n \geq 2$. *Suppose that* $A(z)$ *has* n *distinct zeros* $z_1, \ldots, z_n$, *and* ζ *is a complex number such that no ratio of two zeros of* $A(z)$ *is equal to* ζ. *Prove that*

$$\sum_{k=1}^{n} \frac{1}{A'(z_k)} \left(\frac{1}{A(\zeta z_k)} + \frac{1}{\zeta A(z_k/\zeta)} \right) = 0$$

11. *Let* V_1, V_2 *and* V_3 *be three subspaces of a vectorial space* V *of dimension* n. *Prove that*

$$\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2n$$

12. *Let* h *be a strictly positive real number and let* f *be a continuous function on* $[a-h, a+h]$ *with finite derivative* $f'(x)$ *for every* x *lying in the interval* $(a-h, a+h)$, *where* $a \in \mathbb{R}$. *Prove that if* $f''(a)$ *exists, then*

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

exists and coincides with $f''(a)$. *Does the converse hold?*

13. *Let* $f_k : \mathbb{R} \to \mathbb{R}$, $1 \leq k \leq n$, *be a set of differentiable and linearly independent functions. Show that there exist a set of* $n-1$ *linearly independent functions among the* $f_1', f_2', \ldots, f_n'$.

14. *Let* R, r *be two strictly positive real numbers and let* k *be a positive integer. Consider the sequence of polynomials* $\{F_n(x)\}_{n>k}$ *defined by*

$$F_n(x) = x^n - R^k x^{n-k} - \sum_{\substack{1 \leq j \leq n \\ j \neq k}} r^j x^{n-j}$$

Prove that every F_n *has a unique positive zero* ξ_n *and also show that the sequence* $\{\xi_n\}_{n>k}$ *is convergent.*

15. *Let* a, b, c, d *be real numbers such that* $a > b \geq c > d > 0$. *If* $ad - bc > 0$, *prove that*

$$\prod_{k=1}^{n} \left(\frac{a^{\binom{n}{k}} - b^{\binom{n}{k}}}{c^{\binom{n}{k}} - d^{\binom{n}{k}}} \right)^k \geq \left(\frac{a^{\frac{2^n}{n+1}} - b^{\frac{2^n}{n+1}}}{c^{\frac{2^n}{n+1}} - d^{\frac{2^n}{n+1}}} \right)^{\binom{n+1}{2}}$$