Proposals

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1. Let $A \in M_{2 \times 2}(\mathbb{C})$. Show that $(\text{tr}A)^2 = \text{tr}(A^2) + 2 \det(A)$.

2. Let $f$ be a continuous, nonconstant, real function, and assume the existence of an $F$ such that $f(x + y) = F[f(x), f(y)]$ for all $x, y \in \mathbb{R}$. Show that $f$ is strictly monotone.

3. Let $A_1, A_2, \ldots, A_n$ be the vertices of a regular $n$–gon $P$ inscribed in a circle $C$ with radius one. Let $d_k$, $1 \leq k \leq n$, be the distances from an arbitrary point $P$ lying on $C$ to the vertices of $P$. Prove that

$$\frac{1}{n} \sum_{k=1}^{n} d_k^2$$

is an integer and determine its value.

4. Let $n$ and $k$ be positive integers such that $\sqrt[n]{k}$ is rational. Show that $n$ is a perfect $k^{th}$ power.

5. Let $G$ be a finite group and $K$ be a normal subgroup of $G$ of order $p$, where $p$ is the smallest prime dividing $|G|$. Prove that $K$ is a subgroup of $Z(G)$, where $Z(G)$ denotes the center of the group.

6. The set $A$ consists of nine positive integers, none of which has a prime divisor larger than six. Prove that $A$ has two elements whose product is a perfect square.

7. Let $A$ be the set of positive integers that do not contain the digit 9 in their decimal expansions. Show that the series

$$\sum_{a \in A} \frac{1}{a}$$

is convergent.

8. Let $u, v, w$ be real numbers and let

$$A(x, y, z) = \sum_{\text{sym}} (ux^3 + vx^2y + wxyz)$$

Suppose that $P(1, 1, 1) \geq 0$, $P(1, 1, 0) \geq 0$ and $P(2, 1, 1) \geq 0$. Prove that $P(a, b, c) \geq 0$, whenever $a, b, c$ are the sides of a triangle.

9. Let $A$ be a $3 \times 3$ real orthogonal matrix with $\det(A) = 1$. Prove that

$$(\text{tr}A - 1)^2 + \sum_{i<j} (a_{ij} - a_{ji})^2 = 4$$
10. Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a polynomial of with complex coefficients of degree $n \geq 2$. Suppose that $A(z)$ has $n$ distinct zeros $z_1, \ldots, z_n$, and $\zeta$ is a complex number such that no ratio of two zeros of $A(z)$ is equal to $\zeta$. Prove that

$$\sum_{k=1}^{n} \frac{1}{A'(z_k)} \left( \frac{1}{A(\zeta z_k)} + \frac{1}{\zeta A(z_k/\zeta)} \right) = 0$$

11. Let $V_1, V_2$ and $V_3$ be three subspaces of a vectorial space $V$ of dimension $n$. Prove that

$$\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2n$$

12. Let $h$ be a strictly positive real number and let $f$ be a continuous function on $[a-h, a+h]$ with finite derivative $f'(x)$ for every $x$ lying in the interval $(a-h, a+h)$, where $a \in \mathbb{R}$. Prove that if $f''(a)$ exists, then

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

exists and coincides with $f''(a)$. Does the converse hold?

13. Let $f_k : \mathbb{R} \to \mathbb{R}$, $1 \leq k \leq n$, be a set of differentiable and linearly independent functions. Show that there exist a set of $n-1$ linearly independent functions among the $f'_1, f'_2, \ldots, f'_n$.

14. Let $R, r$ be two strictly positive real numbers and let $k$ be a positive integer. Consider the sequence of polynomials $\{F_n(x)\}_{n>0}$ defined by

$$F_n(x) = x^n - R^k x^{n-k} - \sum_{1 \leq j \leq n \atop j \neq k} r^j x^{n-j}$$

Prove that every $F_n$ has a unique positive zero $\xi_n$ and also show that the sequence $\{\xi_n\}_{n>0}$ is convergent.

15. Let $a, b, c, d$ be real numbers such that $a > b \geq c > d > 0$. If $ad - bc > 0$, prove that

$$\prod_{k=1}^{n} \left( \frac{a^{\frac{1}{n}}}{c^{\frac{1}{n}}} - \frac{b^{\frac{1}{n}}}{d^{\frac{1}{n}}} \right)^k \geq \left( \frac{a^{\frac{2}{n+1}} - b^{\frac{2}{n+1}}}{c^{\frac{2}{n+1}} - d^{\frac{2}{n+1}}} \right)^{(n+1)/2}$$