# *Problems*

This section offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, Room 301-A, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu.*

# Proposals for Olympiads

***1***. Consider the positive integers

$$S_d = 1 + d + d^2 + \cdots + d^{2006},$$

where $d \in \{0, 1, 2, \ldots, 9\}$. Find the last digit of the number

$$S_0 + S_1 + S_2 + \cdots + S_9.$$

***2***. Let $x, y, z$ be positive real numbers. Prove that

$$\sqrt[6]{\frac{xy}{z^2}} + \sqrt[6]{\frac{yz}{x^2}} + \sqrt[6]{\frac{zx}{y^2}} \geq 3.$$

**3**. Let $a, b$ and $c$ be the sides of an acute triangle $ABC$ with semi-perimeter $s$. Prove that

$$\left(\frac{a}{s}\right)^3 \sec A + \left(\frac{b}{s}\right)^3 \sec B + \left(\frac{c}{s}\right)^3 \sec C \geq \frac{16}{9}.$$

**4**. If $x < y < z$, solve in $\mathbb{R}$:

$$\begin{aligned} y(1-x^2) &= 2x \\ z(1-y^2) &= 2y \\ x(1-z^2) &= 2z \end{aligned}$$

**5**. Let $N = \{a_1, a_2, \cdots, a_{n-1}, a_n\}$ be a set of positive real numbers. For all natural numbers $k$, define $S_k$ as the sum of all multiplications of $k$ different elements from the set $N$. Prove that $S_k \cdot S_{n-k} \geq \binom{N}{k}^2 \cdot a_1 \cdot a_2 \cdots \cdot a_{n-1} \cdot a_n$.

**6**. Let $a, b$ and $c$ be positive real numbers such that $ab+bc+ca = 1$. Prove that

$$a\sqrt{b^2+c^2+bc} + b\sqrt{a^2+c^2+ac} + c\sqrt{a^2+b^2+ab} \geq \sqrt{3}.$$

**7**. Let $x, y, z$ be real numbers such that $1 \leq x \leq y \leq z$. Prove that

$$\frac{1}{8}(x+y)^{x+y-z}(y+z)^{y+z-x}(z+x)^{z+x-y} \geq \left(\frac{3}{2}\right)^{x+y+z-3} x^x y^y z^z.$$

**8**. For all positive integers $n, k$ determine the value of

$$\prod_{j=1}^{n}\left(\sum_{l=0}^{k}(-1)^{\ell}\binom{k}{\ell}\cos^{\ell}\frac{2\pi j}{n+1}\right).$$

**9**. (a) Prove that $0 < a_1 < a_2 < \ldots < a_n$ are in arithmetical progression if and only if

$$(a_1 - a_1)(a_3 - a_2)\ldots(a_n - a_{n-1}) = \left(\frac{a_n - a_1}{n-1}\right)^{n-1}$$

(b) Prove that $0 < b_1 < b_2 < \ldots < b_n$ are in geometrical progression if and only if

$$\frac{b_1}{b_n}\left(\frac{b_2}{b_1} + \frac{b_3}{b_2} + \ldots + \frac{b_n}{b_{n-1}}\right)^{n-1} = (n-1)^{n-1}$$

**10**. Find all positive integers $N = \overline{ab0ab}$ such that each $N$ is the product of four consecutive integers.

**11**. If the roots $a, b, c$ of the equation $z^3 + \alpha z^2 + \beta z + 1 = 0$, $(\alpha, \beta, \in \mathbb{C})$ are distinct and nonzero complex numbers, show that

$$\frac{a^2+1}{a^2(b-a)(a-c)} + \frac{b^2+1}{b^2(a-b)(b-c)} + \frac{c^2+1}{c^2(a-c)(c-b)} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

**12**. Find all functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ satisfy

$$f\Big(g(x)\Big) = x^2,$$
$$g\Big(f(x)\Big) = x^3$$

**13**. Let $z_1, z_2, z_3$ be the roots of equation

$$az^3 - az^2 + (a+1)z - (a-1) = 0, \quad (a \in \mathbb{R} - \{0, 1\})$$

Determine the set of values of $a$ for which

$$\frac{z_1}{z_2+z_3}+\frac{z_2}{z_3+z_1}+\frac{z_3}{z_1+z_2}$$

is negative.

***14***. Let $\alpha, \beta$ and $\gamma$ be the angles of triangle $ABC$. Prove that

$$(\csc^2\alpha+\csc^2\beta+csc^2\gamma)(1+\cos\alpha\cos\beta\cos\gamma)\geq\frac{9}{2}.$$

***15***. Solve the following system of equations

$$\begin{aligned} x+y+z+t &= 4 \\ xy+yz+zx+t(x+y+z) &= -1 \\ x^3+y^3+z^3+t^3 &= 28 \\ x^2y^2+y^2z^2+z^2x^2+t^2(x^2+y^2+z^2) &= 105 \end{aligned}$$

***16***. Let $a, b$ and $c$ be the sides of an acute triangle $ABC$. Prove that

$$\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\geq 6\left[\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right)\right]^{1/3}$$

***17***. Two tangents are drawn from the point $P(u, v)$ to the endpoints $(x_1, y_1)$ and $(x_2, y_2)$ of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that $P$ is on the corresponding directrix.

***18***. By definition, a triangle is perfect if its side measures are positive integers and if its perimeter numerically equals its area. Show that any two perfect triangles have the same in-radius.

***19***. Given real numbers $a, b$, and $c$ such that

$$\begin{aligned} a+b+c &= 0 \\ a^2+b^2+c^2 &= 10 \\ a^5+b^5+c^5 &= 10 \end{aligned}$$

Find the value of $a^7+b^7+c^7$.

***20***. Find all real valued function $f$, such that for all real $x, y$

$$f(y) \geq f\left(\frac{x+y}{2}\right) + |x-y|.$$

***21***. Let $x > 1$ be a non-integer number. Prove that

$$\left(\frac{x+\{x\}}{[x]} - \frac{[x]}{x+\{x\}}\right) + \left(\frac{x+[x]}{\{x\}} - \frac{\{x\}}{x+[x]}\right) > \frac{9}{2},$$

where $[x]$ and $\{x\}$ represents the entire and fractional part of $x$.

(Mediterranean Olympiad 2007)