Proposals

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1. *Compute the following expressions:*

$$\cos 1^\circ + \cos 2^\circ + \ldots + \cos 178^\circ + \cos 179^\circ$$

and

$$(1 + \tan 1^\circ)(1 + \tan 2^\circ) \ldots (1 + \tan 44^\circ)$$

2. *Prove that* $2^{2008} - 1$ *is multiple of* 15 *and write the polynomial* $x^{5020} + x^{1004} + 1$ *as product of two polynomials with integer coefficients.*

3. *Prove that, given* n *squares, one can always using only compasses, straight edge and scissors, to construct with them a large square.*

4. *Let* a, b *be nonnegative real numbers such that* $ab \geq a^3 + b^3$. *Prove that* $a + b \leq 1$.

5. *Let* ABC *be an isosceles triangle with* $|AB| = |AC|$. *The height corresponding to the vertex* B *meets the side* AC *in* M. *Prove that*

$$1 + \frac{|AM|}{|MC|} = 2\left(\frac{|AB|}{|BC|}\right)^2$$

6. *Let* $p_1, p_2, \ldots, p_n$ *be* n *people with different ages. Place them in order, so that each one is either older than all the preceding or is younger than all the preceding. In how many ways can this be done?*

7. *Let* ABC *be an acute triangle of perimeter one. Prove that*

$$\sqrt{a \sin A} + \sqrt{b \sin B} + \sqrt{c \sin C} = \sqrt{\sin A + \sin B + \sin C}$$

8. *Let M be a point in the plane of triangle ABC. Prove that*

$$MA^2 + MB^2 + MC^2 \geq \frac{1}{3}(AB^2 + BC^2 + CA^2)$$

When does equality occur?

9. *Let n, m be positive integers. Prove that*

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{m}\right)^{m+1}$$

10. *Let $a_1, a_2, \ldots, a_{13}$ be distinct real numbers. Prove that there exists two of them, say a and b, such that*

$$0 < \frac{a - b}{1 + ab} \leq 2 - \sqrt{3}$$

11. *Let $x, y, z \in [1, +\infty)$. Prove that*

$$\frac{x}{x^2 + yz} + \frac{y}{y^2 + zx} + \frac{z}{z^2 + xy} \leq \frac{3}{2}$$

12. *Let u be the real root of the equation $x^3 - 3x^2 + 5x - 17 = 0$, and let v the real root of the equation $x^3 - 3x^2 + 5x + 11 = 0$. Find $u + v$.*

13. *Prove that for all natural number $n \geq 3$ there exist distinct positive integers $a_1, a_2, \ldots, a_n$ such that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} = 1$.*

14. *Let a, b, c be real numbers such that $abc > 0$. Solve the system of equations*

$$\left.\begin{aligned} x^2 &= a + (y - z)^2, \\ y^2 &= b + (z - x)^2, \\ z^2 &= c + (x - y)^2. \end{aligned}\right\}$$

15. *Let M be a point on the side BC of triangle ABC. Prove that*

$$\left(27\, AM^2 + 8\, BC^2\right)\left(\frac{1}{AB^2\, MC} + \frac{1}{AC^2\, MB}\right) > \frac{108}{BC}$$

16. *Let a, b, c be the lengths of the sides of a triangle ABC with heights h_a, h_b and h_c respectively. Prove that*

$$\prod_{cyclic}\left(\frac{h_a}{h_b+h_c}\right)^{1/3} \le \frac{1}{6}\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

17. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^2\left(\frac{1}{a}+\frac{1}{c}\right)}+\frac{1}{b^2\left(\frac{1}{b}+\frac{1}{a}\right)}+\frac{1}{c^2\left(\frac{1}{c}+\frac{1}{b}\right)} \ge \frac{3}{2}$$

18. *Equation $x^3 - 2x^2 - x + 1 = 0$ has three real roots $a > b > c$. Find the value of $ab^2 + bc^2 + ca^2$.*

19. *Find all real solutions of the equation*

$$2^x + 3^x + 5^x + 7^x = 17^x$$

20. *A set of n circular coins, each of radius r lie without overlapping on a circular table of radius R in such a way that no other coins can be placed on the table without overlapping. Prove that*

$$\frac{1}{2}\left(\frac{R}{r}-1\right) \le \sqrt{n} \le \frac{R}{r}$$

21. *Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \ge 2$. Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.*

22. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{(\sqrt{a}+\sqrt{b})^4}{a+b}+\frac{(\sqrt{b}+\sqrt{c})^4}{b+c}+\frac{(\sqrt{c}+\sqrt{a})^4}{c+a} \ge 24$$

23. *Let a, b, c be the lengths of the sides of a triangle ABC with area one. Let P be any point interior to $\triangle ABC$ and d_a, d_b, d_c represents the* *distances from P to the sides BC, CA and AB, respectively. Prove that*

$$\left(\frac{1}{ad_a}\right)^2 + \left(\frac{1}{bd_b}\right)^2 + \left(\frac{1}{cd_c}\right)^2 \geq \frac{27}{4}$$

24. *Find all triplets (x, y, z) of positive real numbers that are solutions of the system of equations*

$$\begin{aligned} x + y + z &= 1, \\ \frac{xy}{xy+z} + \frac{yz}{yz+x} + \frac{zx}{zx+y} &= \frac{3}{4}. \end{aligned}$$