Problems

This section offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, Room 211-A, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu*.

Proposals for Olympiads

1. Let $\sigma(1), \sigma(2), \ldots, \sigma(2009)$, be an arbitrary permutation of the numbers $1, 2, \ldots, 2009$. Show that the number

$$(\sigma(1)-1)(\sigma(2)-2)\ldots(\sigma(2009)-2009)$$

is even.

2. Find all real functions f such that

$$(x^2+2)f(x)+f(1-x)=x^2+1$$

for all real number x.

3. Determine all points P in the plane of the convex quadrilateral $ABCD$ for which the four numbers $PA^2+PB^2+CD^2$, $PB^2+PC^2+DA^2$, $PC^2+PD^2+AB^2$ and $PD^2+PA^2+BC^2$ have the same value.

4. Determine the set of positive integers n for which the number 10^n+1 is a multiple of 11.

5. Find all triplets (x, y, z) of real numbers for which

$$\sqrt[4x-y^2-2]{\sqrt[4y-z^2-2]{\sqrt[4z-x^2-2]{41x+43y+44z}}}$$

is a positive integer and determine its value.

6. Let $\mathcal{D}$ be a disk of radius 1 and let $P_1, P_2, \cdots, P_n$ be n points lying in the disk and being one of them its center. For any point P_j let x_j be the smallest distance from P_j to some other P_i with $i \neq j$. Prove that

$$x_1^2+x_2^2+\cdots+x_n^2 \leq 9$$

7. Let n be a positive integer and let d by any divisor of $2n^2$. Prove that n^2+d cannot be a perfect square.

8. Let X, Y, Z be points on the respective sides BC, CA, AB of the triangle ABC. If the segments AX, BY, CZ meet in a point P, show that

$$\left(1+\frac{XC}{XB}+\frac{YA}{YC}\right)^{-1}+\left(1+\frac{YA}{YC}+\frac{ZB}{ZA}\right)^{-1}+\left(1+\frac{ZB}{ZA}+\frac{XC}{XB}\right)^{-1} \leq 1$$

9. Compute the following sum

$$\frac{1}{2}\binom{n+1}{n}+\frac{1}{4}\binom{n+2}{n}+\ldots+\frac{1}{2^{n-1}}\binom{2n-1}{n}+\frac{1}{2^n}\binom{2n}{n}$$

10. Let n be a positive integer. Show that the fraction

$$\frac{2010n+7}{1206n+4}$$

is irreducible.

11. Let X, Y, Z be the points where the height, the bisector and the median drawn from vertex A meet the circumcircle $\mathcal{C}(O, R)$ of $\triangle ABC$. Construct the triangle ABC.

12. Find all sequences of integer numbers $\{a_n\}_{n\geq 0}$ such that $a_1 = 1$ and $na_{n-1} + (n-1)a_n = 0$ for all $n \geq 1$.

13. Let x_1, x_2, x_3 be the roots of the equation $x^3 - 2x^2 - x + 1 = 0$. Find the equation which roots are $y_1 = x_1 + \dfrac{1}{x_1}$, $y_2 = x_2 + \dfrac{1}{x_2}$ and $y_3 = x_3 + \dfrac{1}{x_3}$.

14. Let a, b be any real numbers. Prove that

$$\sqrt{a^2 + b^2 + 6a - 2b + 10} + \sqrt{a^2 + b^2 - 6a + 2b + 10} \geq 2\sqrt{10}$$

When does equality occur?

15. Let a, b, c, d be positive real numbers. Prove that

$$\frac{12}{a+b+c+d} \leq \frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{a+d} + \frac{1}{b+c} + \frac{1}{b+d} + \frac{1}{c+d}$$

16. Let x be the real number given by

$$x = 0.123456789101112 \cdots 998999,$$

where the digits are obtained by writing the positive integers 1 trough 999 in order. Find the 2010^{th} digit to the right of the decimal point.

17. Let $a = \underbrace{111\cdots1}_{2010\ 1's}$ and $b = 1\underbrace{00\cdots0}_{2009\ 1's}5$ be positive integers. Find $\sqrt{ab+1}$.

18. If $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \pi/2$, then show that

$$\tan\alpha_1 < \frac{\sin\alpha_1 + \sin\alpha_2 + \cdots + \sin\alpha_n}{\cos\alpha_1 + \cos\alpha_2 + \cdots + \cos\alpha_n} < \tan\alpha_n$$

19. Let a, b, c be positive real numbers. Prove that

$$\frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} + \frac{(a+b)^2}{ab} \geq 12$$

20. Let a, b be positive numbers. Prove that

$$\frac{a}{b(3a+b)} + \frac{b}{a(3b+a)} \geq \frac{4ab}{(a+b)^3}$$

21. Let $a_1, a_2, \ldots, a_7$ be distinct real numbers. Prove that there exists two of them, say a and b, such that

$$0 < \frac{a-b}{1+ab} \leq \frac{\sqrt{3}}{3}$$

22. Let x, y, z be positive real numbers. Prove that

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12$$

23. Let a, b, c be positive numbers. Prove that

$$(a^a b^b c^c)^2 \left(a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)}\right)^3 \geq 27$$

24. Let Let a, b, c be real numbers such that $0 < a, b, c \leq 1$. Prove that

$$\sqrt[3]{\left(\frac{1}{a} + b\right)^3 + \left(\frac{1}{b} + c\right)^3 + \left(\frac{1}{c} + a\right)^3} \geq 2\sqrt[3]{3}$$

25. A bag contains 1000 balls numbered from 1 to 1000. Suppose that 501 balls are chosen from the bag. Prove that there are two of them such that the number of one divides the number of the other.

26. Let n be a positive integer. Show that

$$(n^3 - n)\left(5^{8n+4} + 3^{4n+2}\right)$$

is a multiple of 1902.

27. If no three diagonals of a convex dodecagon meet at the same point inside the dodecagon, into how many line segments are the diagonals divided by their intersections?

28. Find all triplets (x, y, z) of real numbers such that

$$\left.\begin{array}{r} xy(x+y-z)=3, \\ yz(y+z-x)=1, \\ zx(z+x-y)=1. \end{array}\right\}$$

29. Let a, b, c denote the lengths of the sides of a triangle, and r its inradius. Prove that

$$\sum_{cyclic} \frac{a^2(b+c)}{(a+b)(a+c)} \leq \frac{abc}{8r^2}$$

30. Let ABC be a scalene triangle (no two sides equal). The medians from A, B, C, meet the circumcircle again at L, M, N, respectively. If $LM = LN$, prove that $2BC^2 = AB^2 + AC^2$.

31. Let p be a fixed prime. Find the dimensions of all rectangles with integers side lengths whose areas are numerically equal to p times their semi-perimeters.

32. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$ and $\widehat{BCD} = \widehat{EFA} = 60°$. Let G and H be two points in the interior of the hexagon such that $\widehat{AGB} = \widehat{DHE} = 120°$. Prove that $AG + GB + GH + DH + HE \geq CF$.

33. Let ABC be an acute triangle. Let K and L be the feet of the altitudes from A and B, respectively. Let M be the midpoint of AB and let H be the orthocentre of triangle ABC. Prove that the bisector of $\widehat{KML}$ bisects the line segment HC.

34. Assume that the set of nonnegative integers is partitioned into a finite number of infinite arithmetics progressions of ratios

$r_1, r_2, \ldots, r_n$ and first terms $a_1, a_2, \ldots, a_n$. Find the value of

$$\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}$$

35. Find all triplets (x, y, z) of real numbers that satisfy

$$\left.\begin{array}{l} e^{x^4} + \ln x = e^{y^2}, \\ e^{y^4} + \ln y = e^{z^2}, \\ e^{z^4} + \ln z = e^{x^2}. \end{array}\right\}$$

36. Let a, b, c be positive real numbers such that $a^{-2}+b^{-2}+c^{-2} = 1$. Prove that

$$\frac{a+b-2}{a^2+b^2-2} + \frac{b+c-2}{b^2+c^2-2} + \frac{c+a-2}{c^2+a^2-2} < 2\sqrt{3}$$

37. Find all positive integers n such that $5^{n-1}+7^{n-1}$ divides 5^n+7^n.

38. In an acute triangle ABC let us denote by a, b, c the values of $\tan A, \tan B, \tan C$, respectively. Prove that

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{a+b}{ab-1} + \frac{b+c}{bc-1} + \frac{c+a}{ca-1}\right) \geq 9$$

39. Let m, n be positive integers and $A(x)$ be a polynomial of degree n whose coefficients are odd numbers. Suppose that $(x-1)^m$ is a factor of $A(x)$. If $m \geq 2^k$ $(k \geq 2,\ k \in \mathbb{N})$, show that $n \geq 2^{k+1} - 1$.

40. Let r be any irrational number. Prove that there exists a positive integer n so that the distance of nr from the closet integer is less that 10^{-10}.

41. Let a, b, c, d be positive real numbers and $f : [a, b] \to [c, d]$ be a function such that $|f(x) - f(y)| \geq |g(x) - g(y)|$, for all $x, y \in [a, b]$, where $g : \mathbb{R} \to \mathbb{R}$ is a given injective function, with $g(a),\ g(b) \in \{c, d\}$. Prove

(i) $f(a) = c$ and $f(b) = d$, or $f(a) = d$ and $f(b) = c$.

(ii) If $f(a) = g(a)$ and $f(b) = g(b)$, then $f(x) = g(x)$ for $a \leq x \leq b$.

42. Let a, b, c be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2 + b^2(c^2+a^2)^2 + a^2(b^2+c^2)^2} \geq \frac{54}{(a+b+c)^2} \frac{(abc)^3}{\sqrt{(ab)^4 + (bc)^4 + (ca)^4}}$$

43. Find all values of m, n, and p such that

$$p^n + 144 = m^2,$$

where m and n are positive integers and p is a prime number.

44. Let a, b, c, d be the roots of $x^4 + 6x^3 + 7x^2 + 6x + 1 = 0$. Find the value of

$$\frac{3-2a}{1+a} + \frac{3-2b}{1+b} + \frac{3-2c}{1+c} + \frac{3-2d}{1+d}$$

45. A circle is inscribed in a triangle ABC. Let MN be the diameter perpendicular to the base AC. Let L be the intersection of BM with AC. Prove that $AN = LC$.

46. The sum of four real numbers is 9 and the sum of their squares is 21. Prove that these four numbers can be labeled as a, b, c, and d so that the inequality $ab - cd \geq 2$ holds.

47. Suppose that 9 points are given inside a square of side 1. Prove that there are three of them, say A, B, C, such that

$$\mathcal{A}(\triangle ABC) \leq \frac{1}{8}$$

48. Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}} + \sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}} + \sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} \leq \sqrt{3}$$