IMC-2008

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed.

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1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a periodic and differentiable function. Prove that for any positive integer* n *there exists a real number* ξ *such that*

$$f(\xi + n) = f(\xi) + nf'(\xi)$$

Solution by the proposer.

First, we recall that if f is periodic of period p then f is bounded. Indeed, $\mathrm{rang}(f) = \mathrm{rang}(f|_{[0,p]})$. Since $[0, p]$ is compact, then f is bounded and its attains a maximum and minimum in $[0, p]$. Let c and d be the points where f attains a maximum and minimum respectively. Since f is differentiable, then $f'(c) = f'(d) = 0$.

Let $g(x) = f(x + n) - f(x) - nf'(x)$. Since $f'(c) = 0$, then $g(c) = f(c + n) - f(c)$ and $g(c) \leq 0$ (maximum at c). Likewise, $g(d) = f(d + n) - f(d) \geq 0$ (minimum at d). Therefore, g has the intermediate value property (prove this) and exists $\xi \in [c, d]$ such that $g(\xi) = 0$. This completes the proof and we are done.

□

2. *Compute the following determinant*

$$\begin{vmatrix} 1 & a_1 & \ldots & a_1^{n-2} & (a_2 + a_3 + \ldots + a_n)^{n-1} \\ 1 & a_2 & \ldots & a_2^{n-2} & (a_1 + a_3 + \ldots + a_n)^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & \ldots & a_n^{n-2} & (a_1 + a_2 + \ldots + a_{n-1})^{n-1} \end{vmatrix}$$

Solution 1 by Maite Peña, Universidad Pontificia de Comillas, Madrid, Spain.

First of all we can consider that if the determinant is a function $f(a_1, a_2, \ldots, a_n)$ then f is a polynomial of degree $\frac{n(n-1)}{2}$. Since $f(a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots, a_n) = 0$ if $a_i = a_j$, it implies that for any $i, j, i \neq j$, $|a_i - a_j|$ divides f. Thus, we will prove now that

$$f(a_1, a_2, \ldots, a_n) = \prod_{1 \leq i < j \leq n} (-1)^{n+1} |a_i - a_j|$$

But it is easily seen that the last column of the matrix can be written as $(S - a_i)^{n-1}$ where S is the sum of the a_i. Then, just developing $(S - a_i)^{n-1}$ we get that the last term is the only one which is not a linear combination of the previous columns, so the determinant is equal to the determinant of the matrix with the same entries in the columns $1, 2, \ldots, n$.

□

Solution 2 by the proposer.

Let $s = a_1 + a_2 + \cdots + a_n$. Notice that the elements of the last column are of the form

$$(s - a_k)^{n-1} = (-a_k)^{n-1} + \sum_{i=0}^{n-2} b_i x_k^i, \quad (1 \leq k \leq n)$$

Now, adding to the last column a linear combination of the previous with coefficients $-b_0, -b_1, \ldots, -b_{n-2}$, namely,

$$-b_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - b_1 \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} - \ldots - b_{n-2} \begin{pmatrix} a_1^{n-2} \\ a_2^{n-2} \\ \vdots \\ a_n^{n-2} \end{pmatrix},$$

we get

$$\begin{vmatrix} 1 & a_1 & \ldots & a_1^{n-2} & (a_2 + a_3 + \ldots + a_n)^{n-1} \\ 1 & a_2 & \ldots & a_2^{n-2} & (a_1 + a_3 + \ldots + a_n)^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & \ldots & a_n^{n-2} & (a_1 + a_2 + \ldots + a_{n-1})^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a_1 & \ldots & a_1^{n-2} & (-a_1)^{n-1} \\ 1 & a_2 & \ldots & a_2^{n-2} & (-a_2)^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & \ldots & a_n^{n-2} & (-a_n)^{n-1} \end{vmatrix} = (-1)^{n-1} \prod_{i>j} (a_i - a_j)$$

and we are done.

$\square$

Also solved (two solutions) by Xavi Ros, UPC, Barcelona, Spain.

3. *Find all real triplets* (x, y, z) *such that*

$$x + y + z = 2,$$

$$2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2} = 6\sqrt[9]{2}.$$

Solution 1 by Xavi Ros, UPC, Barcelona, Spain.

Applying AM-GM inequality, we have

$$2\sqrt[9]{2} = \frac{1}{3}\left(2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2}\right) \geq \sqrt[3]{2^{x+y+z+x^2+y^2+z^2}}.$$

But since

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0,$$

then

$$3(x^2 + y^2 + z^2) \geq (x + y + z)^2,$$

and

$$x^2 + y^2 + z^2 \geq \frac{4}{3},$$

with equality iff $x = y = z$.
Hence, we have

$$2\sqrt[9]{2} \geq \sqrt[3]{2^{x+y+z+x^2+y^2+z^2}} \geq \sqrt[3]{2^{2+\frac{4}{3}}} = 2\sqrt[9]{2},$$

and as the equality holds iff $x = y = z$, then the unique solution is

$$x = y = z = \frac{2}{3}.$$

$\square$

Solution 2 by the proposer.

Taking into account the well known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

we have

$$x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) \geq 4 - 2(x^2+y^2+z^2)$$

from which immediately follows

$$x^2+y^2+z^2 \geq \frac{4}{3}$$

Applying GM-AM inequality, we have

$$2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2} \geq 3\sqrt[3]{2^{x^2+y^2+z^2+2}} \geq 3\sqrt[3]{2^{2+\frac{4}{3}}} = 6\sqrt[9]{2}$$

Equality holds when $x+y^2 = y+z^2 = z+x^2 = 10/3$. From the preceding we get $x^2+y^2+z^2 = 4/3$ that jointly with the equation $x+y+z = 2$ yields $x^2+y^2+z^2+2(xy+yz+zx) = 4$ and $xy+yz+zx = x^2+y^2+z^2 = 4/3$, from which $x=y=z=2/3$ is the only solution and we are done.

□

Solution 3 by Maite Peña, Universidad Pontificia de Comillas, Madrid, Spain.

We know that $f(x) = 2^x$ is a convex function, so if we apply Jensen's inequality we get

$$\frac{2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2}}{3} = 32^{\frac{10}{9}} \geq 32^{2+x^2+y^2+z^2}$$

Thus, we have $\frac{4}{3} \geq x^2+y^2+z^2$. Using mean's inequality we get

$$\frac{x+y+z}{3} = \frac{2}{3} \leq \sqrt{\frac{x^2+y^2+z^2}{2}}$$

That is, $\frac{4}{3} \leq x^2+y^2+z^2$, so $\frac{4}{3} = x^2+y^2+z^2$, and this happens if and only if $x=y=z=\frac{2}{3}$.

□

4. *Prove that*

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx < \frac{\pi^2}{12}.$$

Solution 1 by the proposer.

We begin with a lemma.

Lema 1 *Let $f : [0,1] \to \mathbb{R}^+$ be a continuous function. Then, the following inequality*

$$\int_0^1 f(x)\,dx \int_0^1 f^2(x)\,dx \leq \int_0^1 f^3(x)\,dx$$

holds.

Proof. First, we observe that $f^2(x)$ and $f^3(x)$ are also continuous. Now, we set $a_k = f(k/n)$ and $b_k = f^2(k/n)$, $(1 \leq k \leq n)$, into Chebyshev's inequality, namely,

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k - \frac{1}{n}\sum_{k=1}^{n} a_k \, \frac{1}{n}\sum_{k=1}^{n} b_k \geq 0$$

and we get

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\, \frac{1}{n}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right) \leq \frac{1}{n}\sum_{k=1}^{n} f^3\left(\frac{k}{n}\right)$$

and the proof follows taking limits when n goes to infinity.

□

Let $\epsilon > 0$ be a sufficiently small positive number and let $f : [\epsilon, 1] \to \mathbb{R}$ be the function defined by $f(x) = \sqrt[3]{\dfrac{\ln(1+x)}{x}}$. Then applying preceding lemma, we have

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\, dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\, dx < \int_0^1 \frac{1}{x}\ln(1+x)\; dx$$

Recalling that the *dilogarithm* function [1] is defined by

$$\text{dilog}(x) = \int_1^x \frac{\ln(t)}{1-t}\; dt$$

and setting $1+t = u$, we have

$$\int_0^1 \frac{1}{t}\ln(1+t)\; dt = -\int_1^2 \frac{\ln(u)}{1-u}\; du = -\text{dilog}(2) + \text{dilog}(0) = \frac{\pi^2}{12}$$

and we are done.

References

[1] Abramowitz, M. and Stegun, I. A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York, 1972.

Solution 2 by Xavi Ros, UPC, Barcelona, Spain.

Let

$$f(x) = \sqrt[3]{\frac{\ln(1+x)}{x}}$$

and

$$g(x) = \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}.$$

First of all, we observe that

$$\frac{1}{2}\int_0^1\int_0^1 [f(x)-f(y)]\,[g(x)-g(y)]\,dx\,dy$$
$$= \int_0^1 f(x)g(x)dx - \int_0^1 f(x)\int_0^1 g(x)dx,$$

but it is easy to check that f and g are strictly decreasing, so

$$\frac{1}{2}\int_0^1\int_0^1 [f(x)-f(y)]\,[g(x)-g(y)]\,dx\,dy > 0,$$

or equivalently,

$$\int_0^1 f(x)dx \int_0^1 g(x)dx < \int_0^1 f(x)g(x)dx.$$

On the other hand,

$$\int_0^1 f(x)g(x)dx = \int_0^1 \frac{\ln(1+x)}{x}dx = \int_0^1 \sum_{n\geq 1}\frac{(-1)^{n-1}x^{n-1}}{n}dx,$$

and as the series is absolutely convergent, then

$$\int_0^1 \sum_{n\geq 1}\frac{(-1)^{n-1}x^{n-1}}{n}dx = \sum_{n\geq 1}\int_0^1 \frac{(-1)^{n-1}x^{n-1}}{n}dx = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n^2}.$$

Finally, taking into account that

$$\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6},$$

we have that

$$\sum_{n\geq 1}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{24},$$

and

$$\sum_{n\geq 1}\frac{1}{(2k+1)^2}=\sum_{n\geq 1}\frac{1}{n^2}-\sum_{n\geq 1}\frac{1}{(2n)^2}=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8},$$

therefore

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^2}=\sum_{k\geq 0}\frac{1}{(2k+1)^2}-\sum_{k\geq 1}\frac{1}{(2k)^2}=\frac{\pi^2}{8}-\frac{\pi^2}{24}=\frac{\pi^2}{12},$$

and

$$\int_0^1 f(x)dx\int_0^1 g(x)dx<\int_0^1 f(x)g(x)dx=\frac{\pi^2}{12},$$

as claimed.

□