Solutions

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed.

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1. *Consider the positive integers*

$$S_d = 1 + d + d^2 + \cdots + d^{2006},$$

where $d \in \{0, 1, 2, \ldots, 9\}$. *Find the last digit of the number*

$$S_0 + S_1 + S_2 + \cdots + S_9.$$

Solution by the proposer.

Denoting by $\mathcal{U}(S_d)$ the last digit of S_d we get $\mathcal{U}(S_0) = \mathcal{U}(1) = 1$ and $\mathcal{U}(S_1) = \mathcal{U}(1 + 2006) = 7$. To obtain the others $\mathcal{U}(S_d)$ we observe that for all $k \in \mathbb{N}$, is $\mathcal{U}(d^{4k} + d^{4k+1} + d^{4k+2} + d^{4k+3}) = 0$, except for $d = 6$. In this case is $\mathcal{U}(6^{5k} + 6^{5k+1} + 6^{5k+2} + 6^{5k+3} + 6^{5k+4}) = 0$. Explicitly,

$$\begin{aligned} \mathcal{U}(S_2) &= \mathcal{U}\left[1 + (2 + 2^2 + 2^3 + 2^4) + \cdots \right. \\ &\quad \left. + (2^{2001} + \ldots + 2^{2004}) + 2^{2005} + 2^{2006}\right] \\ &= \mathcal{U}(1 + 0 + \cdots + 0 + 2 + 4) = 7, \end{aligned}$$

$$\begin{aligned} \mathcal{U}(S_3) &= \mathcal{U}\left[1 + (3 + 3^2 + 3^3 + 3^4) + \cdots + (3^{2001} + \ldots \right. \\ &\quad \left. + 3^{2004}) + 3^{2005} + 3^{2006}\right] \\ &= \mathcal{U}(1 + 0 + \cdots + 0 + 3 + 9) = 3, \end{aligned}$$

$$\begin{aligned} \mathcal{U}(S_4) &= \mathcal{U}\left[1 + (4 + 4^2 + 4^3 + 4^4) + \cdots + (4^{2001} + \ldots \right. \\ &\quad \left. + 4^{2004}) + 4^{2005} + 4^{2006}\right] \\ &= \mathcal{U}(1 + 0 + \cdots + 0 + 4 + 6) = 1, \end{aligned}$$

$$\begin{aligned}\mathcal{U}(S_5) &= \mathcal{U}\left[1+(5+5^2+5^3+5^4)+\dots+(5^{2001}+\dots\right.\\&\left.+5^{2004})+5^{2005}+5^{2006}\right]\\&= \mathcal{U}(1+0+\dots+0+5+5)=1,\end{aligned}$$

$$\begin{aligned}\mathcal{U}(S_6) &= \mathcal{U}\left[1+(6+6^2+6^3+6^4+6^5)+\dots+(6^{2001}+\dots\right.\\&\left.+6^{2004}+6^{2005})+3^{2006}\right]\\&= \mathcal{U}(1+0+\dots+0+0+6)=7,\end{aligned}$$

$$\begin{aligned}\mathcal{U}(S_7) &= \mathcal{U}\left[1+(7+7^2+7^3+7^4)+\dots+(7^{2001}+\dots\right.\\&\left.+7^{2004})+7^{2005}+7^{2006}\right]\\&= \mathcal{U}(1+0+\dots+0+7+9)=7,\end{aligned}$$

$$\begin{aligned}\mathcal{U}(S_8) &= \mathcal{U}\left[1+(8+8^2+8^3+8^4)+\dots+(8^{2001}+\dots\right.\\&\left.+8^{2004})+8^{2005}+8^{2006}\right]\\&= \mathcal{U}(1+0+\dots+0+8+4)=3,\end{aligned}$$

$$\begin{aligned}\mathcal{U}(S_9) &= \mathcal{U}\left[1+(9+9^2+9^3+9^4)+\dots+(9^{2001}+\dots\right.\\&\left.+9^{2004})+9^{2005}+9^{2006}\right]\\&= \mathcal{U}(1+0+\dots+0+9+1)=1.\end{aligned}$$

Therefore,

$$\mathcal{U}(S_0+S_1+S_2+\dots+S_9)=\mathcal{U}(1+7+7+3+1+1+7+7+3+1)=8$$

and we are done. $\square$

2. *Let x, y, z be positive real numbers. Prove that*

$$\sqrt[6]{\frac{xy}{z^2}}+\sqrt[6]{\frac{yz}{x^2}}+\sqrt[6]{\frac{zx}{y^2}}\geq 3.$$

Solution by Xavier Ros, UPC, Barcelona, Spain.

Applying rearrangement's inequality, we have

$$\sqrt[6]{\frac{xy}{z^2}}+\sqrt[6]{\frac{yz}{x^2}}+\sqrt[6]{\frac{zx}{y^2}}=\begin{bmatrix}\sqrt[6]{\frac{x}{z}} & \sqrt[6]{\frac{y}{x}} & \sqrt[6]{\frac{z}{y}}\\ \sqrt[6]{\frac{y}{z}} & \sqrt[6]{\frac{z}{x}} & \sqrt[6]{\frac{x}{y}}\end{bmatrix}\geq\begin{bmatrix}\sqrt[6]{\frac{x}{z}} & \sqrt[6]{\frac{y}{x}} & \sqrt[6]{\frac{z}{y}}\\ \sqrt[6]{\frac{z}{x}} & \sqrt[6]{\frac{x}{y}} & \sqrt[6]{\frac{y}{z}}\end{bmatrix}=3;$$

and we are done.

□

Solution 2 by the proposer

Taking into account the well known inequalities between arithmetic, geometric and harmonic means (AM-GM-HM), we have

$$\begin{aligned}\frac{x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}}{3} &\geq \sqrt[3]{xyz\sqrt{x^2y^2z^2}} = \sqrt[3]{x^2y^2z^2} \\ &\geq \frac{3}{\dfrac{1}{x\sqrt{yz}}+\dfrac{1}{y\sqrt{zx}}+\dfrac{1}{z\sqrt{xy}}}\end{aligned}$$

From the preceding immediately follows

$$\left(\frac{1}{x\sqrt{yz}}+\frac{1}{y\sqrt{zx}}+\frac{1}{z\sqrt{xy}}\right) \geq \frac{3}{\sqrt[3]{x^2y^2z^2}}$$

and

$$\left(\frac{1}{x\sqrt{yz}}+\frac{1}{y\sqrt{zx}}+\frac{1}{z\sqrt{xy}}\right)\sqrt[3]{x^2y^2z^2} \geq 3,$$

or equivalently,

$$\frac{\sqrt[6]{yz}}{\sqrt[3]{x}}+\frac{\sqrt[6]{zx}}{\sqrt[3]{y}}+\frac{\sqrt[6]{xy}}{\sqrt[3]{z}} \geq 3.$$

Notice that equality holds when $x = y = z$ and the proof is complete.

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